

Discretization of Liouville type nonautonomous equations

Ismagil Habibullin¹

Ufa Institute of Mathematics, Russian Academy of Science,

Chernyshevskii Str., 112, Ufa, 450077, Russia

and

Bashkir State University, Z.Validi str. 32, Ufa, 450076, Russia

Natalya Zheltukhina

Department of Mathematics, Faculty of Science,

Bilkent University, 06800, Ankara, Turkey

Abstract

The problem of integrable discretization of Liouville type hyperbolic PDE is studied. The method of discretization preserving characteristic integrals is adopted to nonautonomous case. An intriguing relation between Darboux integrable differential-difference equations and the Guldberg-Vessiot-Lie problem of describing all ODE, possessing fundamental solution systems is observed.

Keywords: semi-discrete chain, Darboux integrability, x -integral, n -integral, continuum limit, discretization, ordinary differential equations, fundamental system of solutions.

1 Introduction

At the present time the problem of discretization of the integrable differential equations is actively studied. In the literature one can find various approaches and techniques used to solve this problem including the Bäcklund transform, the Hamiltonian structure, symmetries, Lax pair, finite gap integration (see [1]-[8]). In [9] we considered the discretization of the Liouville type partial differential equations preserving the structure of one of the integrals, and we constructed the semi-discrete analogues for some equations found by E. Goursat [10]. However, semi-discrete analogues were not found for nonautonomous differential equations. Moreover, in [9] we did not study the continuum limit equations of the chains obtained by the discretization.

In the present paper we improved our discretization method preserving integrability and applied it to nonautonomous cases as well. We also discuss continuum limit equations for

¹e-mail: habibullinismagil@gmail.com

some particular semi-discrete analogues obtained via the discretization. It is rather surprising that discretization of a given Liouville type PDE found by some formal manipulations after evaluation of the continuum limit for vanishing of the grid parameter ε arrives at just the same PDE.

One can see that autonomous hyperbolic equations of Liouville type $u_{xy} = f(u, u_x, u_y)$ found by E. Goursat in [10] have nontrivial autonomous integrals of minimal orders. In the recent paper [11] the authors presented a class of discrete autonomous equations possessing both nontrivial integrals of minimal orders depending on independent discrete variables. The existence of such examples, showing that the class of discrete equations has more complicated structure, stimulated our interest to the discretization problem.

Let us begin with the necessary definitions. We consider semi-discrete chains of the form

$$\frac{d}{dx}t(n+1, x) = f(x, n, t(n, x), t(n+1, x), \frac{d}{dx}t(n, x)), \quad (1)$$

where unknown function $t = t(n, x)$ depends on discrete and continuous variables n and x respectively. We use the following notations throughout the paper:

$$t_k = t(n+k, x), \quad k \in \mathbb{Z}, \quad t_{[m]} = \frac{d^m}{dx^m}t(n, x), \quad m \in \mathbb{N}.$$

Denote by D and D_x the shift operator and the operator of the total derivative with respect to x correspondingly:

$$Dh(n, x) = h(n+1, x), \quad D_x h(n, x) = \frac{d}{dx}h(n, x).$$

Definition 1.1 Functions I and F , depending on $x, n, \{t_k\}_{k=-\infty}^{\infty}, \{t_{[m]}\}_{m=1}^{\infty}$, are called respectively n - and x -integrals of (1), if $DI = I$ and $D_x F = 0$.

Any function depending on n only, is an x -integral, and any function, depending on x only, is an n -integral. Such integrals are called trivial integrals. One can show that any n -integral I does not depend on variables $t_m, m \in \mathbb{Z} \setminus \{0\}$, and any x -integral F does not depend on variables $t_{[m]}, m \in \mathbb{N}$.

Definition 1.2 Chain (1) is called Darboux integrable if it admits a nontrivial n -integral and a nontrivial x -integral.

Chains (1) are semi-discrete analogues of the well-studied hyperbolic equations

$$u_{xy} = g(x, y, u, u_x, u_y). \quad (2)$$

Darboux integrable equations (2) possessing nontrivial x -integral $W(x, y, u, u_y, u_{yy})$ and y -integral $\bar{W}(x, y, u, u_x, u_{xx})$ of order 2 are completely described by the following Theorem.

Theorem 1.3 (see [10]) *Any equation (2), for which there exist second order x - and y -integrals, after the change of variables $x \rightarrow X(x)$, $y \rightarrow Y(y)$, $u \rightarrow U(x, y, u)$, can be reduced to one of the kind:*

- (1) $u_{xy} = e^u$, $\bar{W} = u_{xx} - 0.5u_x^2$, $W = u_{yy} - 0.5u_y^2$;
- (2) $u_{xy} = e^y u_y$, $\bar{W} = u_x - e^u$, $W = \frac{u_{yy}}{u_y} - u_y$;
- (3) $u_{xy} = e^u \sqrt{u_y^2 - 4}$, $\bar{W} = u_{xx} - 0.5u_x^2 - 0.5e^{2u}$, $W = \frac{u_{yy} - u_y^2 + 4}{\sqrt{u_y^2 - 4}}$;
- (4) $u_{xy} = u_x u_y \left(\frac{1}{u-x} - \frac{1}{u-y} \right)$, $\bar{W} = \frac{u_{xx}}{u_x} - \frac{2u_x}{u-x} + \frac{1}{u-x}$, $W = \frac{u_{yy}}{u_y} - \frac{2u_y}{u-y} + \frac{1}{u-y}$;
- (5) $u_{xy} = \psi(u)\beta(u_x)\bar{\beta}(u_y)$, $(\ln\psi)'' = \psi^2$, $\beta\beta' = -u_x$, $\bar{\beta}\bar{\beta}' = -u_y$,
 $\bar{W} = \frac{u_{xx}}{\beta(u_x)} - \psi(u)\beta(u_x)$, $W = \frac{u_{yy}}{\bar{\beta}(u_y)} - \psi(u)\bar{\beta}(u_y)$;
- (6) $u_{xy} = \frac{\beta(u_x)\bar{\beta}(u_y)}{u}$, $\beta\beta' + c\beta = -u_x$, $\bar{\beta}\bar{\beta}' + c\bar{\beta} = -u_y$,
 $\bar{W} = \frac{u_{xx}}{\beta} - \frac{\beta}{u}$, $W = \frac{u_{yy}}{\bar{\beta}} - \frac{\bar{\beta}}{u}$;
- (7) $u_{xy} = -2\frac{\sqrt{u_x u_y}}{x+y}$, $\bar{W} = \frac{u_{xx}}{\sqrt{u_x}} + 2\frac{\sqrt{u_x}}{x+y}$, $W = \frac{u_{yy}}{\sqrt{u_y}} + 2\frac{\sqrt{u_y}}{x+y}$;
- (8) $u_{xy} = \frac{1}{(x+y)\beta(u_x)\bar{\beta}(u_y)}$, $\beta' = \beta^3 + \beta^2$, $\bar{\beta}' = \bar{\beta}^3 + \bar{\beta}^2$,
 $\bar{W} = u_{xx}\beta(u_x) - \frac{1}{(x+y)\beta(u_x)}$, $W = u_{yy}\bar{\beta}(u_y) - \frac{1}{(x+y)\bar{\beta}(u_y)}$.

Throughout the paper we shortly call the eight equations from Theorem 1.3 as the Goursat list. Note that the work [10] contains also equations for which the minimal order integrals are of the order higher than two.

In [9] we made a discretization of equations (2) preserving the structure of y -integrals in each of eight classes from Theorem 1.3.

Theorem 1.4 (see [9]) Below is the list of equations (1) possessing the given n -integral I :

given n - integral	the corresponding chain	
$I = t_{xx} - 0.5t_x^2$	$t_{1x} = t_x + Ce^{0.5(t+t_1)}, C = \text{Const}$	(1*)
$I = t_x - e^t$	$t_{1x} = t_x - e^t + e^{t_1}$	(2*a)
$I = \frac{t_{xx}}{t_x} - t_x$	$t_{1x} = K(t, t_1)t_x$, where $K_t K^{-1} + K_{t_1} = K - 1$	(2*b)
$I = t_{xx} - 0.5t_x^2 - 0.5e^{2t}$	$t_{1x} = t_x + \sqrt{e^{2t} + Re^{t+t_1} + e^{2t_1}}, R = \text{Const}$	(3*a)
$I = \frac{t_{xx} - t_x^2 + 4}{\sqrt{t_x^2 - 4}}$	$t_{1x} = (1 + Re^{t+t_1})t_x + \sqrt{R^2 e^{2(t+t_1)} + 2Re^{t+t_1}} \sqrt{t_x^2 - 4}$	(3*b)
$I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x}$	$t_{1x} = \frac{(t_1+L)(t_1-x)}{(t+L)(t-x)}t_x$, $L = \text{Const}$	(4*)
$I = \frac{t_{xx}}{\beta(t_x)} - \psi(t)\beta(t_x)$ $(\ln\psi)'' = \psi^2, \beta\beta' = -t_x$	$\beta(t_x) = it_x$ and $t_{1x} = K(t, t_1)t_x$, where $K_t + K K_{t_1} + K^2\psi(t_1) - K\psi(t) = 0$	(5*)
$I = \frac{t_{xx}}{\beta(t_x)} - \frac{\beta(t_x)}{t}$, $\beta\beta' + c\beta = -t_x$	$\beta(t_x) = Rt_x$, and $t_{1x} = K(t, t_1)t_x$, where $\frac{K_t}{K} + K_{t_1} = \frac{R^2(tK - t_1)}{tt_1}$	(6*)
$I = \frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x+R}$	$t_{1x} = \left(\sqrt{t_x} + \frac{C}{x+R}\right)^2$, $R = \text{Const}, C = \text{Const}$	(7*)
$I = \beta(t_x)t_{xx} - \frac{1}{(x+R)\beta(t_x)}$, $\beta' = \beta^3 + \beta^2$	$\beta(t_x) = -1$ and $t_{1x} = t_x + \frac{t_1 - t + C}{x+R}$	(8*)

Our discretization method in [9], where n -integrals are functions not depending on n , did not provide semi-discrete equations for each function $\beta(t_x)$ in three cases, namely cases 5, 6 and 8. Also, in cases 4 and 7, where y -integrals depend on x and y , the obtained in [9] semi-discrete chains did not have the corresponding continuous limit equations.

In the present paper we allow n -integral and function f explicitly depend on n , and with this slight modification in the discretization method we again study cases 1, 2, 3, 4, 5 and 7. In cases 5, 6 and 8 n -integrals depend on functions β that are solutions of some differential equations. Only in case 5 these functions β can be written explicitly, and using this exact form for an n -integral we find a new semi-discrete equation, and present it in the following theorem. Thus below we give a semi-discrete version of the fifth equation in the Goursat list.

Theorem 1.5 Semi-discrete equation (1) possessing a minimal order n -integral $I = \frac{t_{xx}}{\beta} - \psi(t, n)\beta$, where $\beta(t_x)$ satisfies $\beta\beta' = -t_x$, is

$$\left(t_{1x} + \sqrt{t_{1x}^2 - M^2}\right)(t_1 + c(n+1)) = \left(t_x - \sqrt{t_x^2 - M^2}\right)(t + c(n)) \quad (3)$$

where $c(n)$ is an arbitrary function of n . It turns out that $\psi(t, n) = -1/(t + c(n))$.

Functions $F = \frac{1}{\psi^2(t_1, n+1)} \left(1 - \frac{\psi^2(t_1, n+1)}{\psi^2(t, n)}\right) \left(1 - \frac{\psi^2(t_1, n+1)}{\psi^2(t_2, n+2)}\right)$ and

$$I = \frac{t_{xx}}{\sqrt{t_x^2 - M^2}} + \frac{\sqrt{t_x^2 - M^2}}{t + c(n)} \text{ are } x\text{- and } n\text{-integrals of (3).}$$

Let us note that in (3) we have $\sqrt{t_x^2 - M^2} = |t_x^2 - M^2|^{1/2} \exp \{i(\text{Arg}(t_x^2 - M^2) + 2\pi k)/2\}$, where $-\pi < \text{Arg}(t_x^2 - M^2) \leq \pi$ and k is a function depending on n by the formula $k(n+1) = 1 + k(n)$, that is, $D\sqrt{t_x^2 - M^2} = -\sqrt{t_{1x}^2 - M^2}$.

In cases 4 and 7, y -integrals depend on the variables x and y . We consider these special nonautonomous cases once again, allowing explicit n -dependence of n -integral and function f , and obtain new semi-discrete chains. In the theorem below we give semi-discrete versions of the equations (4) and (7) from the Goursat list.

Theorem 1.6 (I) *Semi-discrete equation (1) possessing an n -integral $I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x}$ is*

$$t_{1x} = \frac{(1 + t_1 M(n))(t_1 - x)}{(1 + t M(n))(t - x)} t_x \quad (4)$$

where $M(n)$ is an arbitrary function of n . Function $F = \frac{(1 + t_2 M(n+1))(t_1 - t)}{(1 + t M(n))(t_1 - t_2)}$ is an x -integral of (4).

(II) *Semi-discrete equation (1) possessing an n -integral $I = \frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x + \varepsilon n}$ is*

$$t_{1x} = (\sqrt{t_x} + \alpha)^2, \quad \alpha = \sqrt{\frac{\varepsilon(t_1 - t) + C(n)}{(x + \varepsilon n)(x + \varepsilon(n+1))}}, \quad (5)$$

where $C(n)$ is an function of n . Function $F = (x + \varepsilon n)\alpha - (x + \varepsilon(n+2))D\alpha$ is an x -integral of (5).

Theorem 1.7 *Below we display continuum limit equations and x -integrals for semi-discrete equations obtained by the discretization method applied to continuous equations from Theorem*

<i>Semi – discrete equation and its x – integral F</i>	<i>Continuum limit equations and x – integrals \tilde{F}</i>	
$t_{1x} = t_x + Ce^{0.5(t+t_1)}, C = \varepsilon$ $F = e^{(t_1-t)/2} + e^{(t_1-t_2)/2}$	$u_{xy} = e^u$ $\lim_{\varepsilon \rightarrow 0} 2\varepsilon^{-2}(2 - F) = u_{yy} - 0.5u_y^2 = \tilde{F}$	A^*
$t_{1x} = t_x - e^t + e^{t_1}$ $F = (e^t - e^{t_2})(e^{t_1} - e^{t_3})(e^t - e^{t_3})^{-1}(e^{t_1} - e^{t_2})^{-1}$	$u_{xy} = e^u u_y$ $\lim_{\varepsilon \rightarrow 0} \frac{12}{\varepsilon^2}(1 - F) = -2\tilde{F}_y + \tilde{F}^2,$ $\tilde{F} = \frac{u_{yy}}{u_y} - u_y$	B^*
$t_{1x} = K(t, t_1)t_x, K = 1 + \varepsilon e^{t_1}$ $F = e^{t-t_1} + \varepsilon e^t$	$u_{xy} = e^u u_x$ $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1}(1 - F) = u_y - e^u = \tilde{F}$	C^*
$t_{1x} = t_x + \sqrt{e^{2t} + Re^{t+t_1} + e^{2t_1}}, R = -2 - 4\varepsilon^2$ $F = \operatorname{arcsinh}(ae^{t_1-t_2} + b) + \operatorname{arcsinh}(ae^{t_1-t} + b)$ $a = (-4\varepsilon^4 - 4\varepsilon^2)^{-1/2}, b = -(1 + 2\varepsilon^2)a$	$u_{xy} = e^u \sqrt{u_y^2 - 4}$ $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1}(-F + 4 \ln 2) = \frac{u_{yy} - 2u_y^2 + 4}{\sqrt{u_y^2 - 4}} = \tilde{F}$	D^*
$t_{1x} = \sqrt{R^2 e^{2(t+t_1)} + 2Re^{t+t_1}} \sqrt{t_x^2 - 4} +$ $(1 + Re^{t+t_1})t_x, R = 2^{-1}\varepsilon^2$ $F = \sqrt{Re^{2t_1} + 2e^{t_1-t}} + \sqrt{Re^{2t_1} + 2e^{t_1-t_2}}$	$u_{xy} = e^u \sqrt{u_x^2 - 4}$ $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2}(4 - \sqrt{2}F) = u_{yy} - \frac{1}{2}(u_y^2 + e^{2u})$	E^*
$t_{1x} = \frac{(1+t_1M(n))(t_1-x)}{(1+tM(n))(t-x)}t_x, M = -\frac{1}{\varepsilon n}$ $F = \frac{(1+t_2M(n+1))(t_1-t)}{(1+tM(n))(t_1-t_2)}$	$u_{xy} = u_x u_y \left(\frac{1}{u-x} + \frac{1}{u-y} \right)$ $\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}((1 + n^{-1})F + 1) = \frac{1-2u_y}{u-y} + \frac{u_{yy}}{u_y}$	F^*
$t_{1x} = (\sqrt{t_x} + \alpha)^2, \alpha = \sqrt{\frac{\varepsilon(t_1-t)}{(x+\varepsilon n)(x+\varepsilon(n+1))}}$ $F = (x + \varepsilon n)\alpha - (x + \varepsilon(n+2))D\alpha$	$u_{xy} = 2\frac{\sqrt{u_x u_y}}{x+y}$ $\lim_{\varepsilon \rightarrow 0} \frac{-F}{\varepsilon^2} = \frac{\sqrt{u_y}}{x+y} + \frac{1}{2} \frac{u_{yy}}{\sqrt{u_y}}$	G^*

In the present paper we concentrate on the discretization method preserving the structure of one of the integrals. However, we would like to stress that this discretization method has a continuum analogue, that we call as “continualization” method preserving integrability. Let us show how this “continualization” works with two examples.

Example 1. Let us find all equations $t_{xy} = f(x, y, t, t_x, t_y)$ possessing a y -integral $I = t_{xx} - 0.5t_x^2$, that is, we are looking for a continuous analogue of semi-discrete chain $t_{1x} = t_x + Ce^{0.5(t+t_1)}$ (case (1*)) preserving the structure of its n -integral. Equality $D_y I = 0$ becomes $t_{xxy} - t_x t_{xy} = 0$. From the equation searched $t_{xy} = f(x, y, t, t_x, t_y)$ we obtain $t_{xxy} = f_x + f_t t_x + f_{t_x} t_{xx} + f_{t_y} f$. Therefore,

$$f_x + f_t t_x + f_{t_x} t_{xx} + f_{t_y} f - t_x f = 0. \quad (6)$$

Evidently, the coefficient before t_{xx} in (6) vanishes, that is $f_{t_x} = 0$. Now collection of the coefficients before t_x in (6) gives $f_t - f = 0$, or $f = A(x, y, t_y)e^t$. We substitute the expression $f = A(x, y, t_y)e^t$ into (6) and get $A_x e^t + A_{t_y} e^{2t} = 0$ which immediately implies $A_x = A_{t_y} = 0$. Therefore, the equation searched is of the form $t_{xy} = A(y)e^t$ which coincides with the Liouville equation up to a point transformation $y \rightarrow \tilde{y} = \int_0^y A(\theta) d\theta$. It is remarkable that usual continuum limit with small $\varepsilon = C > 0$ approaching zero gives the same answer: equation $(t_{1x} - t_x)/\varepsilon = e^{0.5(t+t_1)}$ becomes the Liouville equation.

Example 2. Let us find all equations $t_{xy} = f(x, y, t, t_x, t_y)$ possessing a y -integral $I = t_{xx} - 0.5t_x^2 - 0.5e^{2t}$, that is we are looking for a continuous analogue of semi-discrete chain $t_{1x} = t_x + \sqrt{e^{2t} + R e^{t+t_1} + e^{2t_1}}$ (case (3*a)) preserving the structure of its n -integral. Equality $D_y I = 0$ becomes $t_{xxy} - t_x t_{xy} - e^{2t} t_y = 0$. We join it with $t_{xyx} = f_x + f_t t_x + f_{t_x} t_{xx} + f_{t_y} f$ and get

$$f_x + f_t t_x + f_{t_x} t_{xx} + f_{t_y} f - t_x f - e^{2t} t_y = 0. \quad (7)$$

The coefficient before t_{xx} in (7) vanishes, that is $f_{t_x} = 0$. Now we collect the coefficients before t_x in (7) and get that $f_t = f$, or $f = C(x, y, t_y)e^t$. We substitute $f = C e^t$ into (7) and have $C_x e^t + C_{t_y} C e^{2t} = t_y e^{2t}$ that immediately implies $C_x = 0$ and $C_{t_y} C = t_y$, or equivalently, $C = \sqrt{t_y^2 + K(y)}$. Therefore, the equation we were looking for is $t_{xy} = e^t \sqrt{t_y^2 - 4K^2(y)}$ which coincides with equation $t_{xy} = e^t \sqrt{t_y^2 - 4}$ up to a point transformation $y \rightarrow \tilde{y} = \int_0^y K(\theta) d\theta$. Note that usual continuum limit with $R = -2 - 4\varepsilon^2$, where ε approaches zero, gives the same answer: equation $t_{1x} = t_x + \sqrt{e^{2t} + R e^{t+t_1} + e^{2t_1}}$ becomes $t_{xy} = e^t \sqrt{t_y^2 - 4}$.

It is widely known that integrable discretization is closely connected with the Bäcklund transform. The next remark is a result of direct verification.

Remark 1.8 *In cases A^* , B^* , C^* , D^* and F^* from Theorem 1.7 the semi-discrete equations realize the Bäcklund transforms of their continuum limit equations. In cases E^* and G^* the discretization differs from the Bäcklund transform.*

We observed that Darboux integrable semi-discrete models of the form (1) have a deep relation with the classical problem of fundamental solution systems for ordinary differential equations going back to S.Lie, E.Vessiot, A.Guldberg. Fundamental systems of solutions for ordinary differential equations were introduced in 1893 (see [12]). Since then they have been studied by several mathematicians (see, e.g. [13]-[14]), more detailed list of references can be found in the survey [15].

Recall that the system of ordinary differential equations

$$\frac{du^i}{dx} = F^i(x, u^1, \dots, u^s), \quad i = 1, \dots, s, \quad (8)$$

possesses a fundamental system of solutions if the general solution to this system of equations may be represented in the form

$$u^i = \phi^i(u_1, \dots, u_m, C_1, \dots, C_s), \quad (9)$$

with the help of a finite number m of arbitrarily chosen particular solutions

$$u_k = (u_k^1, \dots, u_k^s), \quad k = 1, \dots, m, \quad (10)$$

and arbitrary constants C_1, \dots, C_s . The remarkable Lie theorem on ordinary differential equations possessing a fundamental system of solutions is the following.

Theorem 1.9 ([12], also see [15]) *The system of equations (8) possesses a fundamental system of equations if they can be written in the form*

$$\frac{du^i}{dx} = T_1(x)\psi_1^i(u) + \dots + T_r(x)\psi_r^i(u) \quad (11)$$

in such a way that the operators

$$X_\alpha = \psi_\alpha^i(u) \frac{\partial}{\partial u^i}, \quad \alpha = 1, \dots, r, \quad (12)$$

form an r -dimensional Lie algebra. Moreover, the number m of necessary particular (fundamental) solutions (10) satisfies the following condition

$$sm \geq r. \quad (13)$$

To apply the well-known Lie Theorem the system should be written in a special form which is not always an easy task. And even when the system is already in a required form and possesses a fundamental system of solutions, it is at all not clear how to find its fundamental system of solutions.

In the present paper we note that the Darboux integrable semi-discrete chains give rise to a class of differential equations possessing fundamental systems of solutions. We present these equations and discuss how to write their fundamental systems of solutions.

The following problem seems to be of interest. Does x -integral of a Darboux integrable semi-discrete chain provide a fundamental system of solutions for the equations $I = p(x)$ with

I being an n -integral of the corresponding chain. The answer is positive for semi-discrete chains studied in the present paper. Below we give an illustrative example.

Example 3. Semi-discrete version of the Liouville equation

$$t_{1x} = t_x + Ce^{0.5(t+t_1)} \quad (14)$$

admits integrals $I = t_{xx} - 0.5t_x^2$ and $F = e^{(t_1-t)/2} + e^{(t_1-t_2)/2}$. Therefore any solution of the equation (14) satisfies two equations: one of them is an ordinary differential equation and the other – ordinary difference equation:

$$t_{xx} - 0.5t_x^2 = p(x), \quad (15)$$

$$e^{(t_1-t)/2} + e^{(t_1-t_2)/2} = c(n), \quad (16)$$

with two appropriately chosen functions $p = p(x)$ and $c = c(n)$ depending on x and n respectively. Exclude t_2 from equation (16) and the equation $e^{(t_2-t_1)/2} + e^{(t_2-t_3)/2} = c(n+1)$ and get the formula

$$t_3 = -2 \log(C_0 e^{-t_0/2} + C_1 e^{-t_1/2})$$

giving general solution of the equation (15) in terms of two arbitrarily chosen particular solutions $t_0 = t_0(x)$ and $t_1 = t_1(x)$ of the same equation. Here $C_0 = -c(n)$, $C_1 = c(n+1)c(n) - 1$. Hence equation (15) possesses a fundamental system of solutions $\{t_0, t_1\}$.

Remark 1.10 (I) Functions F and $\tilde{F} = DF$ in cases A^* , D^* , E^* , F^* , G^* are x -integrals of order 2. Therefore, $D_x F(x, t, t_1, t_2) = 0$ and $D_x \tilde{F}(x, t_1, t_2, t_3) = 0$. Hence,

$$\begin{cases} F(x, t, t_1, t_2) = c(n), \\ \tilde{F}(x, t_1, t_2, t_3) = c(n+1). \end{cases}$$

One can solve this system of two equations, express t_3 in terms of x , $c(n)$, $c(n+1)$, t , t_1 , and obtain the fundamental system of solutions for the equations

$$I(x, n, t, t_x, t_{xx}) = p(x)$$

for the corresponding n -integrals I .

(II) Any solution t_3 of the equation $I = p(x)$, where $I = t_x - e^t$ is an n -integral from case B^* , satisfies

$$e^{t_3} = \frac{c_0 e^t (e^{t_1} - e^{t_2}) + e^{t_1} (e^t - e^{t_2})}{(e^t - e^{t_2}) + c_0 (e^{t_1} - e^{t_2})},$$

where t , t_1 and t_2 are three particular solutions of the equation $t_x - e^t = p(x)$, and c_0 is an arbitrary constant.

(III) Any solution t_2 of the equation $I = p(x)$, where $I = t_{xx}t_x^{-1} - t_x$ is an n -integral from case C^* , satisfies

$$e^{-t_2} = c_0 + c_1 e^{-t},$$

where t is some solution of the equation $t_{xx}t_x^{-1} - t_x = p(x)$, and c_0, c_1 are arbitrary constants.

We prove Theorems 1.5 and 1.6, and present the proof of Theorem 1.7 and part I of Remark 1.10 in two special cases F^* and G^* . Other cases from Theorem 1.7 and part I of Remark 1.10 can be proved in a similar way. We also discuss the proof of parts II and III of Remark 1.10.

2 Discretization of the fifth equation from the Goursat list

2.1 Discretization. Proof of Theorem 1.5

Let us consider all chains $t_{1x} = f(x, n, t, t_1, t_x)$ with n -integral $I = \frac{t_{xx}}{\beta} - \psi(t, n)\beta$, where $\beta(t_x)$ satisfies $\beta\beta' = -t_x$. We have, $\beta(t_x) = \alpha(n)\sqrt{M^2 - t_x^2}$, where $\alpha(n) = \pm 1$. To shorten the formulas below we will write $\psi(t)$ instead of $\psi(t, n)$, $\psi_1(t_1)$ instead of $\psi(t_1, n+1)$, and $\psi_2(t_2)$ instead of $\psi(t_2, n+2)$. Equality $DI = I$ implies

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{\beta(f)} - \psi_1(t_1)\beta(f) = \frac{t_{xx}}{\beta(t_x)} - \psi(t)\beta(t_x). \quad (17)$$

We compare the coefficients before t_{xx} and have

$$\frac{\alpha(n+1)f_{t_x}}{\sqrt{M^2 - f^2}} = \frac{\alpha(n)}{\sqrt{M^2 - t_x^2}}.$$

There are two possibilities: (a) $\alpha(n)\alpha(n+1) = 1$ and (b) $\alpha(n)\alpha(n+1) = -1$. In case (a) we have

$$f + \sqrt{f^2 - M^2} = L(x, n, t, t_1)(t_x + \sqrt{t_x^2 - M^2}), \quad (18)$$

or

$$f = \frac{L(x, n, t, t_1)}{2}(t_x + \sqrt{t_x^2 - M^2}) + \frac{M^2}{2L(x, n, t, t_1)(t_x + \sqrt{t_x^2 - M^2})}. \quad (19)$$

In case (b) we have

$$f + \sqrt{f^2 - M^2} = \frac{V(x, n, t, t_1)}{t_x + \sqrt{t_x^2 - M^2}}, \quad (20)$$

or

$$f = \frac{V(x, n, t, t_1)}{2} \frac{1}{t_x + \sqrt{t_x^2 - M^2}} + \frac{M^2}{2V(x, n, t, t_1)} (t_x + \sqrt{t_x^2 - M^2}). \quad (21)$$

We differentiate (18) with respect to x and get $f_x = \frac{L_x}{L} \sqrt{f^2 - M^2}$. Similarly, one can get $f_t = \frac{L_t}{L} \sqrt{f^2 - M^2}$ and $f_{t_1} = \frac{L_{t_1}}{L} \sqrt{f^2 - M^2}$. Substitute these expressions into (17) and get

$$\begin{aligned} & \frac{L_x}{L} + \frac{L_t}{L} t_x + \frac{L_{t_1}}{L} \left(\frac{L}{2} (t_x + \sqrt{t_x^2 - M^2}) + \frac{M^2}{2L(t_x + \sqrt{t_x^2 - M^2})} \right) \\ &= \psi_1(t_1) \left(\frac{L}{2} (t_x + \sqrt{t_x^2 - M^2}) + \frac{M^2}{2L(t_x + \sqrt{t_x^2 - M^2})} \right) - \psi(t) \sqrt{t_x^2 - M^2}. \end{aligned}$$

Compare the coefficients before t_x^0 , t_x , $\sqrt{t_x^2 - M^2}$, $(t_x + \sqrt{t_x^2 - M^2})^{-1}$ in the last equality and have

$$L_x = 0, \quad \frac{L_t}{L} + \frac{L_{t_1}}{2} = \frac{\psi_1(t_1)L}{2}, \quad \frac{L_{t_1}}{2} = \frac{\psi_1(t_1)L}{2} - \psi(t), \quad \frac{L_{t_1}}{L} = -\psi_1(t_1).$$

We solve these equations and see that $L = \frac{\psi(t)}{\psi_1(t_1)}$, where $\psi(t)$ satisfies $\frac{\psi'(t)}{\psi^2(t)} = \frac{\psi'_1(t_1)}{\psi_1^2(t_1)}$, or $\psi(t) = \frac{1}{At+B(n)}$. Hence, $L = \frac{t_1+C(n+1)}{t+C(n)}$. Therefore, the semi-discrete chain in case (a) is

$$\frac{t_{1x} + \sqrt{t_{1x}^2 - M^2}}{t_1 + C(n+1)} = \frac{t_x + \sqrt{t_x^2 - M^2}}{t + C(n)}, \quad (22)$$

where $C(n)$ is an arbitrary function of n . One can see that $I = \frac{t_x + \sqrt{t_x^2 - M^2}}{t + C(n)}$ is an n -integral of (22). It means that in case (a) we obtain not only the chain that has a given n -integral of order 2, but also an n -integral of order 1.

Let us study case (b). We differentiate (20) with respect to x and get $f_x = \frac{V_x}{V} \sqrt{f^2 - M^2}$. Similarly, one can get $f_t = \frac{V_t}{V} \sqrt{f^2 - M^2}$ and $f_{t_1} = \frac{V_{t_1}}{V} \sqrt{f^2 - M^2}$. Substitute these expressions into (17) and get

$$\begin{aligned} & \frac{V_x}{V} + \frac{V_t}{V} t_x + \frac{V_{t_1}}{V} \left(\frac{V}{2} \frac{1}{t_x + \sqrt{t_x^2 - M^2}} + \frac{M^2}{2V} (t_x + \sqrt{t_x^2 - M^2}) \right) \\ &= \psi_1(t_1) \left(\frac{V}{2} \frac{1}{t_x + \sqrt{t_x^2 - M^2}} - \frac{M^2}{2V} (t_x + \sqrt{t_x^2 - M^2}) \right) + \psi(t) \sqrt{t_x^2 - M^2}. \end{aligned}$$

By comparing the coefficients before t_x^0 , t_x , $\sqrt{t_x^2 - M^2}$, $(t_x + \sqrt{t_x^2 - M^2})^{-1}$ we obtain

$$V_x = 0, \quad \frac{V_t}{V} + \frac{M^2 V_{t_1}}{2V^2} + \psi_1(t_1) \frac{M^2}{2V} = 0, \quad V_{t_1} = \psi(t_1)V, \quad \frac{V_{t_1} M^2}{2V^2} + \psi_1(t_1) \frac{M^2}{2V} = \psi(t).$$

We solve these equations and have $V = \frac{M^2 \psi_1(t_1)}{\psi(t)}$, where $\psi(t)$ satisfies $\psi' = \psi^2$. Hence, $V = \frac{M^2(t+C(n))}{t_1+C(n+1)}$. Therefore, the semi-discrete equation becomes

$$(t_{1x} + \sqrt{t_{1x}^2 - M^2}) = \frac{M^2(t + C(n))}{t_1 + C(n+1)} \frac{1}{t_x + \sqrt{t_x^2 - M^2}}.$$

It can be rewritten as

$$(t_1 + C(n+1))(t_{1x} + \sqrt{t_{1x}^2 - M^2}) = (t + C(n))(t_x - \sqrt{t_x^2 - M^2}). \quad (23)$$

One can check that

$$I = \frac{t_{xx}}{\sqrt{t_x^2 - M^2}} + \frac{\sqrt{t_x^2 - M^2}}{t + C(n)}$$

is indeed an integral of (23). Note that in this case $D\sqrt{t_x^2 - M^2} = -\sqrt{t_{1x}^2 - M^2}$.

2.2 Finding of x -integral

Let us find an x -integral $F = F(x, n, t, t_1, t_2)$ of (23). Equation (23) can be also rewritten as

$$t_{1x} = \frac{t_x}{2} \left(\frac{\psi(t)}{\psi_1(t_1)} + \frac{\psi_1(t_1)}{\psi(t)} \right) + \frac{\sqrt{t_x^2 - M^2}}{2} \left(\frac{\psi_1(t_1)}{\psi(t)} - \frac{\psi(t)}{\psi_1(t_1)} \right). \quad (24)$$

Then

$$t_{2x} = \frac{t_x}{2} \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} + \frac{\psi_2(t_2)\psi(t)}{\psi_1^2(t_1)} \right) + \frac{\sqrt{t_x^2 - M^2}}{2} \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} - \frac{\psi(t)\psi_2(t_2)}{\psi_1^2(t_1)} \right). \quad (25)$$

Then $D_x F = 0$ becomes $F_x + F_t t_x + F_{t_1} t_{1x} + F_{t_2} t_{2x} = 0$, or

$$\begin{aligned} & F_x + F_t t_x + F_{t_1} \left(\frac{t_x}{2} \left(\frac{\psi(t)}{\psi_1(t_1)} + \frac{\psi_1(t_1)}{\psi(t)} \right) + \frac{\sqrt{t_x^2 - M^2}}{2} \left(\frac{\psi_1(t_1)}{\psi(t)} - \frac{\psi(t)}{\psi_1(t_1)} \right) \right) \\ & + F_{t_2} \left(\frac{t_x}{2} \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} + \frac{\psi_2(t_2)\psi(t)}{\psi_1^2(t_1)} \right) + \frac{\sqrt{t_x^2 - M^2}}{2} \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} - \frac{\psi(t)\psi_2(t_2)}{\psi_1^2(t_1)} \right) \right) = 0. \end{aligned}$$

Compare the coefficients before t_x^0 , t_x and $\sqrt{t_x^2 - M^2}$ in the last equality and have the following system of equations

$$\begin{cases} F_x = 0 \\ 2F_t + \left(\frac{\psi(t)}{\psi_1(t_1)} + \frac{\psi_1(t_1)}{\psi(t)} \right) F_{t_1} + \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} + \frac{\psi_2(t_2)\psi(t)}{\psi_1^2(t_1)} \right) F_{t_2} = 0 \\ \left(\frac{\psi_1(t_1)}{\psi(t)} - \frac{\psi(t)}{\psi_1(t_1)} \right) F_{t_1} + \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} - \frac{\psi(t)\psi_2(t_2)}{\psi_1^2(t_1)} \right) F_{t_2} = 0 \end{cases} \quad (26)$$

that can be rewritten as

$$\begin{cases} F_x = 0 \\ \left(\frac{\psi_1(t_1)}{\psi(t)} - \frac{\psi(t)}{\psi_1(t_1)} \right) F_t + \left(\frac{\psi_2(t_2)}{\psi_1(t_1)} - \frac{\psi_1(t_1)\psi(t)}{\psi_2(t_2)} \right) F_{t_2} = 0 \\ \left(\frac{\psi_1(t_1)}{\psi(t)} - \frac{\psi(t)}{\psi_1(t_1)} \right) F_{t_1} + \left(\frac{\psi_1^2(t_1)}{\psi(t)\psi_2(t_2)} - \frac{\psi(t)\psi_2(t_2)}{\psi_1^2(t_1)} \right) F_{t_2} = 0 \end{cases} \quad (27)$$

With new variables $\tilde{t} = t + C(n)$, $\tilde{t}_1 = t_1 + C(n+1)$ and $\tilde{t}_2 = t_2 + C(n+2)$ we have $\psi(\tilde{t}) = -1/\tilde{t}$, and the system of equations becomes

$$\begin{cases} \frac{\tilde{t}_1^2 - \tilde{t}^2}{\tilde{t}} F_{\tilde{t}} + \frac{\tilde{t}_2^2 - \tilde{t}^2}{\tilde{t}_2} F_{\tilde{t}_2} = 0 \\ (\tilde{t}_1^2 - \tilde{t}^2) F_{\tilde{t}_1} + \frac{\tilde{t}_1^4 - \tilde{t}^2 \tilde{t}_2^2}{\tilde{t}_1 \tilde{t}_2} F_{\tilde{t}_2} = 0 \end{cases}$$

In new variables $u = \tilde{t}$, $w = \tilde{t}_1$ and $v = (\tilde{t}_1^2 - \tilde{t}^2)(\tilde{t}_2^2 - \tilde{t}^2)$ the last system can be rewritten as

$$\begin{cases} F_u = 0 \\ F_w + 2vw^{-1}F_v = 0 \end{cases}$$

that implies $F = \frac{v}{w^2}$. We can write F in variables \tilde{t} , \tilde{t}_1 and \tilde{t}_2 as $F = \tilde{t}_1^{-2}(\tilde{t}_1^2 - \tilde{t}^2)(\tilde{t}_2^2 - \tilde{t}^2)$.

In old variables t , t_1 and t_2 function F is given as

$$F = -\psi^{-2}(t_1, n+1) \left(1 - \frac{\psi^2(t_1, n+1)}{\psi^2(t, n)}\right) \left(1 - \frac{\psi^2(t_1, n+1)}{\psi^2(t_2, n+2)}\right) \text{ then. } \square$$

3 Discretization of the fourth equation from the Goursat list

3.1 Discretization. Proof of Theorem 1.6, Part I.

We consider semi-discrete equations $t_{1x} = f(x, n, t, t_1, t_x)$ with n -integral

$$I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x} \quad (28)$$

From $DI = I$ we get

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{f} - \frac{2f}{t_1 - x} + \frac{1}{t_1 - x} = \frac{t_{xx}}{t_x} - \frac{2t_x}{t-x} + \frac{1}{t-x} \quad (29)$$

By comparing the coefficients in (29) before t_{xx} we obtain $f_{t_x}/f = 1/t_x$, or $f = t_x K$, where K is some function depending on x , n , t and t_1 . Substitute $f = t_x K$ into (29) and find

$$\frac{K_x t_x + K_t t_x^2 + K_{t_1} K t_x^2}{K t_x} - \frac{2K t_x}{t_1 - x} + \frac{1}{t_1 - x} = -\frac{2t_x}{t-x} + \frac{1}{t-x} \quad (30)$$

Compare the coefficients before t_x and t_x^0 in (30) and get

$$\frac{K_t}{K} + K_{t_1} = \frac{2K}{t_1 - x} - \frac{2}{t-x} \quad (31)$$

$$\frac{K_x}{K} = \frac{1}{t-x} - \frac{1}{t_1 - x} \quad (32)$$

We solve (32) and have $K = C(t_1 - x)/(t - x)$, where C is some function depending on n , t and t_1 . Substitute this expression for K into (31) and obtain

$$\frac{C_t}{C}(t - x) + C_{t_1}(t_1 - x) = C - 1 \quad (33)$$

By comparing the coefficients before x and x^0 in (33) we get the system of equations

$$\begin{cases} \frac{C_t}{C} + C_{t_1} = 0, \\ \frac{C_t}{C}t + C_{t_1}t_1 = C - 1 \end{cases}$$

whose solution is $C = (1 + M(n)t_1)/(1 + M(n)t)$. Thus, equation $t_{1x} = f(x, n, t, t_1, t_x)$ possessing n -integral (28) is

$$t_{1x} = \frac{(1 + M(n)t_1)(t_1 - x)}{(1 + M(n)t)(t - x)}t_x, \quad (34)$$

where $M(n)$ is an arbitrary function depending on n only.

3.2 Evaluation of the x -integral

Let us find an x -integral of equation (34) of minimal order if it exists. First, assume that equation (34) possesses an x -integral $F(x, n, t, t_1)$ of the first order. The equality $D_x F(x, n, t, t_1) = 0$ can be rewritten as

$$F_x + F_t t_x + F_{t_1} \frac{(1 + M(n)t_1)(t_1 - x)}{(1 + M(n)t)(t - x)} t_x = 0 \quad (35)$$

By comparing the coefficients before t_x^0 and t_x we get

$$F_x = 0 \quad (36)$$

and

$$F_t + F_{t_1} \frac{(1 + M(n)t_1)(t_1 - x)}{(1 + M(n)t)(t - x)} = 0 \quad (37)$$

We differentiate equation (37) with respect to x , use (36), and get a contradictory equality

$$\frac{\partial}{\partial x} \left\{ \frac{t_1 - x}{t - x} \right\} = 0.$$

It means that equation (34) does not possess an x -integral $F(x, n, t, t_1)$ of the first order.

Now let us see whether equation (34) possesses an x -integral $F(x, n, t, t_1, t_2)$ of the second order. Since $D_x F = 0$ then

$$\begin{aligned} & F_x + F_t t_x + F_{t_1} \frac{(1 + M(n)t_1)(t_1 - x)}{(1 + M(n)t)(t - x)} t_x \\ & + F_{t_2} \frac{(1 + M(n+1)t_2)(t_2 - x)(1 + M(n)t_1)(t_1 - x)}{(1 + M(n+1)t_1)(t_1 - x)(1 + M(n)t)(t - x)} t_x = 0 \end{aligned} \quad (38)$$

By comparing the coefficients before t_x^0 and t_x we get

$$F_x = 0 \quad (39)$$

and

$$F_t + F_{t_1} \frac{(1 + M(n)t_1)(t_1 - x)}{(1 + M(n)t)(t - x)} + F_{t_2} \frac{(1 + M(n+1)t_2)(t_2 - x)(1 + M(n)t_1)}{(1 + M(n+1)t_1)(1 + M(n)t)(t - x)} = 0 \quad (40)$$

We differentiate equation (40) with respect to x and get

$$F_{t_1}(t_1 - t) + F_{t_2} \frac{(1 + M(n+1)t_2)(t_2 - t)}{(1 + M(n+1)t_1)} = 0 \quad (41)$$

One can check that the system of partial differential equations (39), (40) and (41) is closed. To solve this system of equations we use the famous Jacobi Method: we first diagonalise the system (that is, we make it normal) and then we do the necessary changes of variables using the first integrals of the equations from the system. The calculations are standard but rather long. That is why we omit these straightforward steps and present an x -integral immediately. It is

$$F(x, n, t, t_1, t_2) = \frac{(1 + M(n+1)t_2)(t_1 - t)}{(1 + M(n)t)(t_1 - t_2)} \quad (42)$$

For the readers familiar with the characteristic rings (see [22], [23], [24]) we would like to note that the existence of a nontrivial x -integral for equation (34) implies that the characteristic ring L_x in x -direction for this equation is of finite dimension. It is not difficult to see that for equation (34) characteristic ring L_x is generated by three vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t} + \frac{(1 + M(n)t_1)(t_1 - x)}{(1 + M(n)t)(t - x)} \frac{\partial}{\partial t_1} + \frac{(1 + M(n+1)t_2)(t_2 - x)(1 + M(n)t_1)}{(1 + M(n+1)t_1)(1 + M(n)t)(t - x)} \frac{\partial}{\partial t_2}, \\ X_3 &= (t_1 - t) \frac{\partial}{\partial t_1} + \frac{(1 + M(n+1)t_2)(t_2 - t)}{(1 + M(n+1)t_1)} \frac{\partial}{\partial t_2}. \end{aligned} \quad (43)$$

In particular, it means that the dimension of L_x for equation (34) is 3.

3.3 Continuum limits. Proof of Theorem 1.7, Case F^*

In semi-discrete equation (34) we rewrite $t(x, n)$ as $u(x, y)$, t_1 as $u(x, y) + \varepsilon u_y(x, y)$, $M(n)$ as $1/R(\varepsilon n) = 1/R(y)$, and get

$$u_x + \varepsilon u_{xy} = \left(\frac{R(y) + u + \varepsilon u_y}{R(y) + u} \right) \left(\frac{u + \varepsilon u_y - x}{u - x} \right) u_x,$$

or

$$u_x + \varepsilon u_{xy} = \left(1 + \frac{\varepsilon u_y}{u + R(y)}\right) \left(1 + \frac{\varepsilon u_y}{u - x}\right) u_x,$$

or

$$u_{xy} = u_x u_y \left(\frac{1}{u - x} + \frac{1}{u + R(y)} \right) + \varepsilon \frac{u_y^2 u_x}{(u - x)(u + R(y))}$$

Now we let ε approach 0 to get continuous equation analogue

$$u_{xy} = \left(\frac{1}{u - x} + \frac{1}{u + R(y)} \right) u_x u_y. \quad (44)$$

Note that after the change of variable $\tilde{y} = -R(y)$ equation (44) becomes

$$u_{x\tilde{y}} = \left(\frac{1}{u - x} + \frac{1}{u - \tilde{y}} \right) u_x u_{\tilde{y}}. \quad (45)$$

In x -integral $\varepsilon^{-1}(1 + (1 + n^{-1})F)$ of semi-discrete equation (34), where F is taken as (42) we substitute u , $u + \varepsilon u_y + 0.5\varepsilon^2 u_{yy}$, $1/R(y)$ and y instead of t , t_1 , $M(n)$ and εn correspondingly, and let ε approach 0 to get its continuous analogue

$$\tilde{F} = -\frac{u_{yy}}{u_y} + \frac{R'(y)}{u + R(y)} + \frac{2u_y}{u + R(y)} \quad (46)$$

Note that continuous equation (44) possesses y -integral (28) and x -integral (46)

3.4 Fundamental solution system for the associated ODE, case F^*

Equation (34) admits the following n - and x - integrals

$$I = \frac{t_{xx}}{t_x} - \frac{2t_x}{t - x} + \frac{1}{t - x},$$

$$F = \frac{(1 + M(n+1)t_2)(t_1 - t)}{(1 + M(n)t)(t_1 - t_2)}$$

Solution t_n of differential equation

$$\frac{t_{xx}}{t_x} - \frac{2t_x}{t - x} + \frac{1}{t - x} = p(x) \quad (47)$$

can be found from the following system of equations

$$\begin{cases} F = c(n) & \rightarrow \frac{(1 + M(n+1)t_2)(t_1 - t)}{(1 + M(n)t)(t_1 - t_2)} = c(n) \\ DF = c(n+1) & \rightarrow \frac{(1 + M(n+2)t_3)(t_2 - t_1)}{(1 + M(n+1)t_1)(t_2 - t_3)} = c(n+1) \end{cases}$$

One can easily solve the system of these two equations and express general solution t_3 of (47) in terms of its arbitrarily chosen particular solutions $t = t(x)$ and $t_1 = t_1(x)$ as follows

$$t_3 = \frac{(t_1 - t)(1 - C_1) + C_1 C_0 t_1(1 + M(n)t)}{(t_1 - t)(C_1 M(n+1) - M(n+2)) + C_1 C_0(1 + M(n)t)},$$

where $C_0 = c(n)$, $C_1 = c(n+1)$. Thus for any choice of the r.h.s. $p(x)$ equation (47) possesses a fundamental solution system.

4 Discretization of the seventh equation from the Gour-sat list

4.1 Discretization. Proof of Theorem 1.6, Part II

Let us consider semi-discrete equations (1) possessing n -integral

$$I = \frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x + \varepsilon n} \quad (48)$$

Since $DI = I$ then

$$\frac{f_x + f_t t_x + f_{t_1} f + f_{t_x} t_{xx}}{\sqrt{f}} + \frac{2\sqrt{f}}{x + \varepsilon(n+1)} = \frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x + \varepsilon n} \quad (49)$$

We compare the coefficients before t_{xx} in (49) and get $f_{t_x}/\sqrt{f} = 1/\sqrt{t_x}$, or $\sqrt{f} = \sqrt{t_x} + L$, where L is some function depending on x, n, t, t_1 . We substitute $f = (\sqrt{t_x} + L)^2$ into (49) and have

$$L_x + L_t t_x + L_{t_1}(\sqrt{t_x} + L)^2 + \frac{\sqrt{t_x} + L}{x + \varepsilon(n+1)} = \frac{\sqrt{t_x}}{x + \varepsilon n}$$

that implies that function $L(x, n, t, t_1)$ satisfies the following three differential equations

$$L_t + L_{t_1} = 0 \quad (50)$$

$$2LL_{t_1} + \frac{1}{x + \varepsilon(n+1)} = \frac{1}{x + \varepsilon n} \quad (51)$$

$$L_x + L^2 L_{t_1} + \frac{L}{x + \varepsilon(n+1)} = 0 \quad (52)$$

Equation (51) gives that

$$L^2 = \left(\frac{1}{x + \varepsilon n} - \frac{1}{x + \varepsilon(n+1)} \right) t_1 + M \quad (53)$$

where M is some function depending on x , n and t . We substitute the expression for L^2 from (53) into the equation (50) rewritten as $LL_t + LL_{t_1} = 0$ and obtain

$$M = \left(\frac{1}{x + \varepsilon(n+1)} - \frac{1}{x + \varepsilon n} \right) t + K$$

where K is some function depending on x and n only. Thus,

$$L^2 = \left(\frac{1}{x + \varepsilon n} - \frac{1}{x + \varepsilon(n+1)} \right) (t_1 - t) + K$$

Substitute this expression for L^2 into the equation (52) multiplied by $2L$ and have

$$K_x = \left(\frac{1}{x + \varepsilon(n+1)} - \frac{1}{x + \varepsilon n} \right) K \quad \rightarrow \quad K = \frac{C(n)}{(x + \varepsilon n)(x + \varepsilon(n+1))},$$

where $C(n)$ is an arbitrary function of n . Therefore,

$$L^2 = \frac{\varepsilon(t_1 - t) + C(n)}{(x + \varepsilon n)(x + \varepsilon(n+1))}$$

and then

$$f(x, n, t, t_1, t_x) = \left(\sqrt{t_x} + \sqrt{\frac{\varepsilon(t_1 - t) + C(n)}{(x + \varepsilon n)(x + \varepsilon(n+1))}} \right)^2 \quad (54)$$

Let us note that one can eliminate function $C(n)$ in (54) by the change of variable $t(x, n) = \tau(x, n) + d(n)$, where $d(n)$ satisfies $\varepsilon(d(n+1) - d(n)) + C(n) = 0$. Equations possessing n -integral (48) become

$$t_{1x} = \left(\sqrt{t_x} + \sqrt{\frac{\varepsilon(t_1 - t)}{(x + \varepsilon n)(x + \varepsilon(n+1))}} \right)^2 \quad (55)$$

4.2 Evaluation of the x -integral

Let us find x -integral of equation (55). Denote by

$$\alpha = \sqrt{\frac{\varepsilon(t_1 - t)}{(x + \varepsilon n)(x + \varepsilon(n+1))}} \quad \beta = D\alpha = \sqrt{\frac{\varepsilon(t_2 - t_1)}{(x + \varepsilon(n+1))(x + \varepsilon(n+2))}} \quad (56)$$

We find an x -integral of the minimal order of equation (55) in the same way as we did for equation (34). We look for function $F(x, n, t, t_1, t_2)$ such that $D_x F = 0$. We have,

$$F_x + F_t t_x + F_{t_1} (t_x + \alpha^2 + 2\sqrt{t_x}\alpha) + F_{t_2} (\sqrt{t_x} + \alpha + \beta)^2 = 0 \quad (57)$$

Compare the coefficients before t_x , $\sqrt{t_x}$ and t_x^0 in (57) and get the following system of equation

$$\begin{cases} F_t + F_{t_1} + F_{t_2} = 0 \\ \alpha F_{t_1} + (\alpha + \beta) F_{t_2} = 0 \\ F_x + \alpha^2 F_{t_1} + (\alpha + \beta)^2 F_{t_2} = 0 \end{cases}$$

that can be rewritten as

$$\begin{cases} F_x + \beta(\alpha + \beta)F_{t_2} = 0 \\ \alpha F_t - \beta F_{t_2=0} \\ \alpha F_{t_1} + (\alpha + \beta)F_{t_2} = 0 \end{cases}$$

One can check that the system is closed and its solution is

$$F = (x + \varepsilon n)\alpha - (x + \varepsilon(n + 2))\beta. \quad (58)$$

4.3 Continuum limits. Proof of Theorem 1.7, Part G^*

In semi-discrete equation (55) we substitute u , $u + \varepsilon u_y$ and y instead of t , t_1 and εn correspondingly, and let ε approach 0 to get its continuous Liouville equation analogue

$$u_{xy} = \frac{2\sqrt{u_x u_y}}{x + y} \quad (59)$$

In x -integral (58) multiplied by $-2\varepsilon^{-2}$ we substitute u , $u + \varepsilon u_y + 0.5\varepsilon^2 u_{yy}$ and y instead of t , t_1 and εn correspondingly, and let ε approach 0 to get its continuous analogue

$$\tilde{F} = \frac{u_{yy}}{\sqrt{u_y}} + \frac{2}{x + y} \quad (60)$$

Note that continuous equation (59) possesses y -integral

$$I = \frac{u_{xx}}{\sqrt{u_x}} + \frac{2\sqrt{u_x}}{x + y}$$

which is a continuous analogue of (48) and x -integral (60)

4.4 Fundamental solution system for the associated ODE, case G^*

As it was shown above in the previous section equation (55) possesses the following n - and x -integrals

$$I = \frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x + \varepsilon n},$$

$$F = (x + \varepsilon n)\sqrt{\frac{\varepsilon(t_1 - t)}{(x + \varepsilon n)(x + \varepsilon(n + 1))}} - (x + \varepsilon(n + 2))\sqrt{\frac{\varepsilon(t_2 - t_1)}{(x + \varepsilon(n + 1))(x + \varepsilon(n + 2))}}$$

Solution t_n of differential equation

$$\frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x + \varepsilon n} = p(x) \quad (61)$$

can be found from the following system of equations

$$\begin{cases} F = c(n) & \rightarrow & \sqrt{t_2 - t_1} = \sqrt{\frac{(x + \varepsilon n)(t_1 - t)}{x + \varepsilon(n + 2)}} - \frac{c(n)}{\sqrt{\varepsilon}} \sqrt{\frac{x + \varepsilon(n + 1)}{x + \varepsilon(n + 2)}} \\ DF = c(n + 1) & \rightarrow & \sqrt{t_3 - t_2} = \sqrt{\frac{(x + \varepsilon(n + 1))(t_2 - t_1)}{x + \varepsilon(n + 3)}} - \frac{c(n + 1)}{\sqrt{\varepsilon}} \sqrt{\frac{x + \varepsilon(n + 2)}{x + \varepsilon(n + 3)}} \end{cases}$$

We solve the system of these two equations and express function t_3 in terms of arbitrary functions t and t_1 as follows

$$t_3 = t_1 + \frac{\gamma_1^2}{\varepsilon}(\gamma\sqrt{\varepsilon(t_1 - t)} + C_0)^2 + \frac{\gamma_2^2}{\varepsilon}((\gamma_1^2(\gamma\sqrt{\varepsilon(t_1 - t)} + C_0) + C_1), \quad (62)$$

where $C_0 = -c(n)$, $C_1 = -c(n + 1)^2$ and

$$\gamma = \sqrt{\frac{x + \varepsilon n}{x + \varepsilon(n + 1)}}, \quad \gamma_1 = D\gamma, \quad \gamma_2 = D\gamma_1$$

Formula (62) shows that solution t_3 of the ODE (61) is expressed through its particular solutions t and t_1 and two arbitrary constants C_0, C_1 .

5 Proof of parts II and III of Remark 1.10

Consider case B^* . Equation $t_x - e^t = p(x)$ satisfies Lie Theorem 1.9 with $T_1(x) = 1$, $\psi_1(t) = e^t$, $T_2(x) = p(x)$, and $\psi_2(t) = 1$. Moreover, vector fields $X_1 = e^t \frac{\partial}{\partial t}$ and $X_2 = \frac{\partial}{\partial t}$ generate a 2-dimensional Lie algebra. Thus, $r = 2$, $s = 1$. Therefore, the number m of necessary particular solutions satisfies $m \geq 2$. It turns out that $m = 3$. Equation $F = c(n)$, where F is an x -integral, becomes

$$\frac{(e^t - e^{t_2})(e^{t_1} - e^{t_3})}{(e^t - e^{t_3})(e^{t_1} - e^{t_2})} = c(n),$$

that implies

$$e^{t_3} = \frac{c(n)e^t(e^{t_1} - e^{t_2}) + e^{t_1}(e^t - e^{t_2})}{(e^t - e^{t_2}) + c(n)(e^{t_1} - e^{t_2})}.$$

Consider case C^* . Equation $t_{xx}t_x^{-1} - t_x = p(x)$ satisfies Lie Theorem 1.9 with $\vec{u} = \{u_1, u_2\}$, $T_1(x) = 1$, $\psi_1^1(\vec{u}) = u_1$, $\psi_1^2(\vec{u}) = u_2^2$, $T_2(x) = p(x)$, and $\psi_2^1(\vec{u}) = 0$, $\psi_2^2(\vec{u}) = u_2$. Moreover, vector fields $X_1 = u_1 \frac{\partial}{\partial u_1} + u_2^2 \frac{\partial}{\partial u_2}$ and $X_2 = u_2 \frac{\partial}{\partial u_2}$ generate a 3-dimensional Lie algebra. Thus, $r = 3$ and $s = 2$. Therefore, the number m of necessary particular solutions satisfies $m \geq 2$. It turns out that $m = 2$. Equations $F = c(n)$ and $DF = c(n + 1)$, where F is an x -integral,

are $e^{t-t_1} + e^t = c(n)$ and $e^{t_1-t_2} + e^{t_1} = c(n+1)$. We express e^{t_1} from the first equation and substitute it in the second equation, and get

$$e^{-t_2} = C_0 + C_1 e^{-t}, \quad (63)$$

where $C_0 = -1 - c(n+1)$ and $C_1 = c(n+1)c(n)$.

6 Conclusion

Darboux integrable equations or equations of Liouville type constitute a very well studied and important in applications subclass of all hyperbolic type PDE. The problem of complete description of this subclass was formulated and partly solved by Goursat in 1899. Since then many authors investigated the problem [16]-[21]. An interesting problem, closely connected with the one mentioned above is the description of all functions which might be integrals of Darboux integrable equations. The results of the present article allow one to conjecture that if the function $F(x, y, u, u_x, u_{xx}, \dots, \frac{d^k u}{dx^k})$ is an integral of a Darboux integrable PDE $u_{xy} = g(x, y, u, u_x, u_y)$ then the ODE

$$F(x, y, u, u_x, u_{xx}, \dots, \frac{d^k u}{dx^k}) = p(x) \quad (64)$$

possesses fundamental solution system for any r.h.s. $p = p(x)$. The conjecture is proved for a majority of the equations from the Goursat list. The method of proof is based on the discretization preserving integrals.

A semi-discrete equation

$$t_{1x} = f(x, n, t, t_1, t_x), \quad (65)$$

is called a discretization preserving integral of the Liouville type PDE

$$u_{xy} = g(x, y, u, u_x, u_y) \quad (66)$$

if two equations (65) and (66) have a common integral $I = I(x, y, u, u_x, u_{xx}, \dots)$ such that

$$D_y I(x, y, u, u_x, u_{xx}, \dots) = 0$$

and

$$(D_n - 1)I(x, n\varepsilon, t, t_x, t_{xx}, \dots) = 0$$

Equation (66) in its turn is called “continualization” of the equation (65). Note that such kind “continualization” does not use any continuum limit procedure, only computations based on the notion of the integral. Surprisingly this “continualization” and the standard algorithm of evaluation of the continuum limit equation for the equations studied in the article give one and the same result. For instance, semi-discrete equation (55) is the discretization of the equation (59) since

$$D_y\left(\frac{u_{xx}}{\sqrt{u_x}} + \frac{2\sqrt{u_x}}{x+y}\right) = 0$$

and

$$(D_n - 1)\left(\frac{t_{xx}}{\sqrt{t_x}} + \frac{2\sqrt{t_x}}{x + \varepsilon n}\right) = 0.$$

Equation (59) is the “continualization” of the equation (55). Formal continuum limit in (55) gives the same answer.

The discretization scheme applied to the Goursat list (Theorem 1.3) generates the similar list of semi-discrete equations which turned out to be also Darboux integrable (see Theorem 1.7). However we have to confess that we failed to find satisfactory discretization for the eighth equation in the Goursat list.

Acknowledgments

This work is partially supported by Russian Foundation for Basic Research (RFBR) grants 13-01-00070-a and 14-01-97008-r-povolzhie-a.

References

- [1] M. Bruschi, D. Levi, and O. Ragnisco, *Discrete version of the nonlinear Schrödinger equation with linearly x-dependent coefficients*, Il Nuovo Cimento A Series 11, 53(1), 21-30 (1979).
- [2] Y. B. Suris, *The problem of integrable discretization: Hamiltonian approach*, Vol. 219. Springer, 2003.
- [3] R. Hirota and K. Kimura, *Discretization of the Euler top*, Journal of the Physical Society of Japan 69 (2000): 627.

- [4] M. Murata, et al., *How to discretize differential systems in a systematic way*, Journal of Physics A: Mathematical and Theoretical 43.31 (2010): 315203.
- [5] J. Moser and A. P. Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Communications in Mathematical Physics 139.2 (1991): 217-243.
- [6] J. Gibbons and B. A. Kupershmidt, *Time discretizations of lattice integrable systems*, Physics Letters A 165.2 (1992): 105-110.
- [7] D. Zakharov, *A discrete analogue of the modified Novikov-Veselov hierarchy*, arXiv.org.nlin.arXiv:0904.3728v2
- [8] V. E. Adler, *On a discrete analog of the Tzitzeica equation*, (arXiv:1103.5139)
- [9] I. T. Habibullin, N. Zheltukhina, and A. Sakieva, *Discretization of hyperbolic type Darboux integrable equations preserving integrability*, J. Math. Phys. 52 (2011), 093507.
- [10] E. Goursat, *Recherches sur quelques équations aux dérivées partielles du second ordre*, Annales de la faculté des Sciences de l'Université de Toulouse 2e série, 1:1 (1899), 3178.
- [11] R. N. Garifullin and R. I. Yamilov, *Generalized symmetry classification of discrete equations of a class depending on twelve parameters*, Journal of Physics A: Mathematical and Theoretical 45.34 (2012): 345205.
- [12] S. Lie, *Vorlesungen über continuerliche Gruppen mit geometrischen und anderen Anwendungen*, Bearbeitet und herausgegeben von Dr. Scheffers, Leipzig: B.G.Teubner (1893).
- [13] E. Vessiot, *Sur les classe d'équations différentielles*, Ann.Sci.Ecole Norm. Sup. (1893), T.10, P.53.
- [14] A. Guldberg, *Sur les équations différentielles ordinaire qui possèdent un système fundamental d'intégrales*, C. r. Acad. sci. Paris (1893), T.116, P.964.
- [15] N. K. Ibragimov, *Group analysis of ordinary differential equations and the invariance principle in mathematical physics (for the 150th anniversary of Sophus Lie)*, Russian Mathematical Surveys, 47(4), 89-156 (1992).

- [16] N. F. Gareeva and A. V. Zhiber, *The second order integrals of the hyperbolic equations and evolutionary equations*, in Proceedings of the International Conference "Algebraic and analytic methods in the theory of the differential equations", 1996, Orel, edited by A.G.Meshkov, pp.39-42.
- [17] A. V. Zhiber and V. V. Sokolov, *Exactly integrable hyperbolic equations of Liouville type*, Russian Mathematical Surveys, 56(1), 61 (2001).
- [18] M. E. Lainé, *Sur une équation de la forme $s = p\phi(x; y; z; q)$ intégrable par la méthode de Darboux*, Comptes rendus, V. 183, 1926, P. 1254-1256.
- [19] E. Vessiot, *Sur les équations aux dérivées partielles du second order, $F(x; y; z; p; q; r; s; t) = 0$, intégrable par la méthode de Darboux*, J. Math. pure appl., V. 18, 1939. P. 1-61.
- [20] E. Vessiot, *Sur les équations aux dérivées partielles du second order, $F(x; y; z; p; q; r; s; t) = 0$, intégrable par la méthode de Darboux*, J. Math. pure appl., V. 21, 1942. P. 1-66.
- [21] O. V. Kaptsov, *On the Goursat classification problem*, Programming and Computer Software 38 (2), 102-104.
- [22] A. V. Zhiber, R.D. Murtazina, I. T Habibullin and A. B. Shabat, *Characteristic Lie rings and nonlinear integrable equations*, – M.-Izhevsk, 2012. – pp. 376 (in Russian).
- [23] I. Habibullin I, N. Zheltukhina and A. Pekcan A, *On the classification of Darboux integrable chains*, J. Math. Phys. 49, 2008, 102702
- [24] I. Habibullin, N. Zheltukhina and A. Pekcan, *Complete list of Darboux integrable chains of the form $t_{1x} = t_x + d(t, t_1)$* , J. Math. Phys. 50, 2009 102710